Problem 1: H and I

Wednesday, October 02, 2013 10:19 AM

First, we need to find the unknown constant c.
The given description for the joint pmf can be expressed using the joint pmf matrix as

$$P_{X,Y} = \begin{array}{c|cc} x \backslash y & 2 & 4 \\ \hline 1 & 3c & 5c \\ 3 & 5c & 7c \end{array} \quad \text{i.e.} \quad \begin{bmatrix} 3c & 5c \\ 5c & 7c \end{bmatrix}$$

Recall that $\sum_x \sum_y p_{X,Y}(x,y) = 1$.

Here, we have

$$3c + 5c + 5c + 7c = 1$$

$$20c = 1$$

$$c = \frac{1}{20}.$$

a) $H(X,Y) = H\left(\left[\frac{3}{20} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{7}{20}\right]\right) = -\frac{3}{20}\log_2\frac{3}{20} - \frac{2}{4}\log_2\frac{1}{4} - \frac{7}{20}\log_2\frac{7}{20}$

≈ 1.9406 bits.

To find $H(X)$ and $H(Y)$, we need p_X and p_Y, respectively.
These can be found from the sums along the rows and columns of $P_{X,Y}$.

$$\begin{array}{c|cc|l} x \backslash y & 2 & 4 & \\ \hline 1 & 3c & 5c & \rightarrow 8c = \frac{8}{20} = \frac{2}{5} \\ 3 & 5c & 7c & \rightarrow 12c = \frac{12}{20} = \frac{3}{5} \\ \hline & \downarrow & \downarrow & \\ & 8c & 12c & \\ & \parallel & \parallel & \\ & 2/5 & 3/5 & \end{array}$$

b) $H(X) = H\left(\left[\frac{2}{5} \quad \frac{3}{5}\right]\right) \approx 0.9710$

c) $H(Y) = H\left(\left[\frac{2}{5} \quad \frac{3}{5}\right]\right) \approx 0.9710$

d) $H(X|Y) = H(X,Y) - H(Y) \approx 0.9697$

e) $H(Y|X) = H(X,Y) - H(X) \approx 0.9697$

f) $I(X;Y) = H(X) + H(Y) - H(X,Y) \approx 0.0013$

Problem 2: H and I

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First, we need to find the unknown constant β
The given description for the joint pmf can be expressed using the joint pmf matrix as

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 3 & 1/15 & 4/15 \\ 4 & 2/15 & \beta \end{array}$$

Recall that $\sum_x \sum_y p_{X,Y}(x,y) = 1.$

Here, we have

$$\frac{1}{15} + \frac{4}{15} + \frac{2}{15} + \beta = 1$$

$$\beta = 1 - \frac{7}{15} = \frac{8}{15}.$$

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc|c} x \backslash y & 1 & 3 & \\ \hline 3 & 1/15 & 4/15 & \rightarrow 5/15 = 1/3 \\ 4 & 2/15 & 8/15 & \rightarrow 10/15 = 2/3 \\ \hline & \downarrow & \downarrow & \\ & 3/15 = 1/5 & 12/15 = 4/5 & \end{array}$$

Notice that $(\mathbf{p}_X)^T (\mathbf{p}_Y) = \mathbf{P}_{X,Y}$; that is $p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$ for all pair (x,y).

Therefore, $X \perp\!\!\!\perp Y$.

a) $H(X,Y) \overset{X \perp\!\!\!\perp Y}{=} H(X) + H(Y) \approx 1.6402$

b) $H(X) \overset{X \perp\!\!\!\perp Y}{=} H([\,1/3 \quad 2/3\,]) \approx 0.9183$

$X \perp\!\!\!\perp Y$

c) $H(Y) = H([1/5 \quad 4/5]) \approx 0.7219$

d) $H(X|Y) = H(X) \approx 0.9183$

$X \perp\!\!\!\perp Y$

e) $H(Y|X) = H(Y) \approx 0.7219$

f) $I(X;Y) = 0$ because $X \perp\!\!\!\perp Y$

Problem 3: C

Wednesday, October 02, 2013 10:51 AM

(a) Symmetric Channel $\Rightarrow C = \log_2|S_Y| - H(\underline{r}) = \log_2 2 - H([\frac{1}{3} \ \frac{2}{3}])$

$$= 1 - 0.9183 \approx 0.0817$$

(b) Note that this is a noisy channel with non overlapping outputs.

So, $C = \log_2|S_X| = \log_2 3 \approx 1.5850$.

(c) $I(X;Y) = \underbrace{H(Y)}_{} - \underbrace{H(Y|X)}_{= H([1/3 \ \ 2/3])}$ ← all rows of Q have the same collection of conditional probabilities.

$$\underline{q} = \underline{p} Q = \begin{bmatrix} p_0 & 1-p_0 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}p_0 & \frac{2}{3}p_0 + \frac{2}{3}(1-p_0) & \frac{1}{3}(1-p_0) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}p_0 & \frac{2}{3} & \frac{1}{3}(1-p_0) \end{bmatrix}$$

Because $H(Y|X)$ is fixed, we need to maximize $H(Y)$.

To do this, note that

$$H(Y) = -\frac{1}{3}p_0 \log_2 \frac{1}{3}p_0 - \frac{2}{3}\log_2\frac{2}{3} - \frac{1}{3}(1-p_0)\log_2\frac{1}{3}(1-p_0).$$

$$(\ln 2) H(Y) = -\frac{1}{3}p_0 \ln\left(\frac{1}{3}p_0\right) - \frac{2}{3}\ln\frac{2}{3} - \frac{1}{3}(1-p_0)\ln\left(\frac{1}{3}(1-p_0)\right).$$

$$\Big\downarrow \frac{d}{dp_0}$$

$$= -\frac{1}{3}\left(\cancel{1} + \ln\left(\frac{1}{3}p_0\right)\right) - \left(-\frac{1}{3}\right)\left(\cancel{1} + \ln\left(1 + \frac{1}{3}(1-p_0)\right)\right)$$

$$= \frac{1}{3}\left(\ln\left(\frac{\cancel{1/3}(1-p_0)}{\cancel{1/3}\,p_0}\right)\right) = \frac{1}{3}\ln\left(\frac{1-p_0}{p_0}\right) = \frac{1}{3}\ln\left(\underbrace{\frac{1}{p_0} - 1}_{}\right)$$

As p_0 increases from 0^+ to 1^-, $\frac{1}{p_0} - 1$ decreases from $+\infty$ to 0^+

$\ln\left(\frac{1}{p_0} - 1\right)$ decreases from $+\infty$ to $-\infty$.

Note: $\frac{d}{dx} f(x)\ln f(x) = f'(x)\ln f(x) + \cancel{f(x)}\frac{1}{\cancel{f(x)}} f'(x)$

$$= (f'(x))(1 + \ln f(x))$$

The derivative is 0 at $p_0 = \frac{1}{2}$. For $p_0 < \frac{1}{2}$, the derivative is > 0;

$\frac{1}{3} \ln\left(\frac{1}{p_0} - 1\right) = 0$

For $p_0 > \frac{1}{2}$, " " < 0.

So, $p_0 = \frac{1}{2}$ is the global maximum.

When $p_0 = 1/2$,

$$H(Y) = H\left(\left[\frac{1}{6} \quad \frac{2}{3} \quad \frac{1}{6}\right]\right).$$

$$\text{So, } C = H\left(\left[\frac{1}{6} \quad \frac{2}{3} \quad \frac{1}{6}\right]\right) - H\left(\left[1/3 \quad 2/3\right]\right)$$

$$= -\frac{2}{6}\log_2\frac{1}{6} - \cancel{\frac{2}{3}\log_2\frac{2}{3}} + \frac{1}{3}\log_2\frac{1}{3} + \cancel{\frac{2}{3}\log_2\frac{2}{3}}$$

$$= \frac{1}{3}\log_2\frac{6}{3} = \frac{1}{3}.$$

(d) The rows of Q are the same. So, $q(y) = Q(y|x)$ which implies $X \perp\!\!\!\perp Y$.

Therefore, $I(X;Y) = 0$ for any input distribution.

Hence, $C = 0$.